Q.1

Let S be a closed surface enclosing a region R in $\mathbb{R}^3$.

(a) Show that the volume enclosed is given by

$$\mathrm{Vol}(R) = \frac{1}{3}\int_S (\vec{r}-\vec{c})\cdot\vec{n}\,da$$

where $\vec{r} = (x, y, z)$ and $\vec{c}$ is a constant vector.

(b) Hence, show that if M = diameter of $S := \max\{|x-y| : x, y \in S\}$

$$\mathrm{Vol}(R) \leq \frac{1}{3} M \,\mathrm{Area}(S).$$

Solution:

(a) Consider a vector field F which has div F = 3. The easiest choice is $F(x,y,z) = (x,y,z) = \vec{r} - \vec{c}$

Then by the divergence theorem, we have

$$\int_R 3\,dV = \int_S (\vec{r}-\vec{c})\cdot\vec{n}\,d\sigma$$

$$3\,\mathrm{vol}(R) = \int_S (\vec{r}-\vec{c})\cdot\vec{n}\,d\sigma$$

$$\mathrm{vol}(R) = \frac{1}{3}\int_S (\vec{r}-\vec{c})\cdot\vec{n}\,d\sigma$$

(b)
$$\mathrm{vol}(R) = \frac{1}{3}\int_S (\vec{r}-\vec{c})\cdot\vec{n}\,d\sigma$$

$$= \frac{1}{3}\left|\int_S (\vec{r}-\vec{c})\cdot\vec{n}\,d\sigma\right|$$

$$\leq \frac{1}{3}\int_S |(\vec{r}-\vec{c})\cdot\vec{n}|\,d\sigma$$

$$\leq \frac{1}{3}\int_S |\vec{r}-\vec{c}||\vec{n}|\,d\sigma \quad \text{(Cauchy-Schwarz)}$$

(Choose $\vec{c}$ to be any fixed pt on S.)

$$\leq \frac{1}{3} M \int_S 1\,d\sigma \qquad (|\vec{n}| = 1)$$

$$= \frac{1}{3} M\,\mathrm{Area}(S)$$

Q.2

Let $\Omega \subseteq \mathbb{R}^3$ be an open set with compact closure.

Suppose $f : \overline{\Omega} \times [0,\alpha] \to \mathbb{R}^3$ be a C^2 function satisfying

$$\begin{cases} \dfrac{\partial f}{\partial t}(x,y,z,t) = c \, \Delta f^m (x,y,z,t), \quad (x,y,z,t) \in \Omega \times [0,\alpha] \\ D_{\vec{n}(x,y,z)} f(x,y,z) = 0 \quad \text{for } (x,y,z,t) \in \partial\Omega \times [0,\alpha] \end{cases}$$

where $\Delta g = \operatorname{div}(\nabla g)$, $c > 0$, $m > 1$ are constants.

Show that $\int_\Omega f \, dV$ is independent of time.

Solution:

$$\frac{d}{dt}\int_{\Omega} f\,dV \overset{(ex.)}{=} \int_{\Omega} \frac{\partial f}{\partial t}\,dV$$

$$= \int_{\Omega} \operatorname{div}(\nabla f^m)\,dV$$

$$= \int_{\partial\Omega} \nabla f^m \cdot n\,d\sigma$$

$$= \int_{\partial\Omega} m f^{m-1}\,\nabla f \cdot n\,d\sigma$$

$$= \int_{\partial\Omega} m f^{m-1}\,D_n f\,d\sigma$$

$$= 0 \quad \text{since } D_n f = 0 \text{ on } \partial\Omega.$$

Q.3

Prove the Stokes' theorem for the following special case:

- $S \cup \partial S$ is the graph of a C^2 function $f : \overline{\mathcal{U}} \to \mathbb{R}^3$ with $f(\partial \mathcal{U}) = \partial S$
- F is a C^1 vector field defined in an open set $\mathcal{U}$ containing $S \cup \partial S$ by $F(x,y,z) = (0,0,P(x,y,z))$.

Solution:

Let $r(t) = (x(t), y(t)), t \in [0,1]$ be a parametrization of $\partial \mathcal{U}$.

Then a parametrization of ∂S is

$$\gamma(t) = \big(x(t), y(t), f(x(t), y(t))\big), \ t \in [0,1].$$

Then

$$\int_{\partial S} F \cdot d\vec{r} = \int_0^1 (P \circ \gamma) \frac{d}{dt}(f(x(t), y(t)))\,dt$$

$$= \int_0^1 (P \circ \gamma)(f_x x' + f_y y')\,dt$$

$$= \int_{\partial \mathcal{U}} P f_x dx + P f_y dy$$

$$= \int_{\mathcal{U}} \big((Pf_y)_x - (Pf_x)_y\big)\,dA$$

$$= \int_{\mathcal{U}} (P_x f_y + P f_{yx} - P_y f_x - P f_{xy})\,dA$$

$$= \int_{\mathcal{U}} (P_x f_y - P_y f_x)\,dA$$

On the other hand, S is parametrized by $\varphi : U \to \mathbb{R}^3$,

$$\varphi(x,y) = (x, y, f(x,y))$$

Then $\varphi_x = (1, 0, f_x)$

$\varphi_y = (0, 1, f_y)$

$$\varphi_x \times \varphi_y = (-f_x, -f_y, 1)$$

$$\nabla \times F = (P_y, -P_x, 0)$$

$$\therefore \int_S (\nabla \times F) \cdot n \, d\sigma$$

$$= \int_U (P_y, -P_x, 0) \cdot (-f_x, -f_y, 1) \, dA$$

$$= \int_U (P_x f_y - P_y f_x) \, dA$$

$$\therefore \int_{\partial S} F \cdot d\vec{r} = \int_S (\nabla \times F) \cdot n \, d\sigma$$